Unit 5
Factorization

EXERCISE 5.1

Factorize

Q1. (i) $2abc - 4abx + 2abd$

$= 2ab(c - 2x + d)$

(ii) $9xy - 12x^2y + 18y^2$

$= 3y\,(3x - 4x^2 + 6y)$

(iii) $-3x^2y - 3x + 9xy^2$

$= -3x(xy + 1 - 3y^2)$

(iv) $5ab^2c^3 - 10a^2b^3c - 20a^3bc^2$

$= 5abc(bc^2 - 2b^2 - 4a^2c$

(v) $3x^3y(x - 3y) - (7x^2y^2(x - 3y)$

$= (x - 3y)\,(3x^3y - 7x^2y^2)$

$= (x - 3y)\,x^2y(3x - 7y)$

$= x^2y(x - 3y)x^2y(3x - 7y)$

(vi) $2xy^3(x^2 + 5) + 8xy^2(x^2 + 5)$

$= (x^2 + 5)\,(2xy^3 + 8xy^2)$

$= (x^2 + 5)\,(2xy^2)(y + 4)$

$= 2xy^2(x^2 + 5)(y + 4)$

Q2. (i) $5ax - 3ay - 5bx + 3by$

$= 5ax - 5bx - 3ay + 3by$

$= 5x(a - b)\,(5x - 3y)$

(ii) $3xy + 2y - 12x - 8$

$= 3xy - 12x + 2y - 8$

$= 3x(y - 4) + 2(y - 4)$

$= (y - 4)\,(3x + 2)$

(iii) $x^3 + 3xy^2 - 2x^2 - 6y^3$

$= x(x^2 + 3y^2) - 2y(x^2 + 3y^2)$

$= (x^2 + 3y^2)\,(x - 2y)$

(iv) $(x^2 - y^2)z + (y^2 - z^2)x$

$= x^2z - y^2z + y^2x - z^2x$

$= x^2z - z^2x + y^2x - y^2z$

$= xz(x - z) + y^2x - y^2z$

$= (x - z)(xz + y^2)$

Q3. (i) $144a^2 + 24a + 1$

$= 144a^2 + 12a + 12a + 1$

$= 12a(12a + 1) + 1(12a + 1)$

$= (12a + 1)(12a + 1) = (12a + 1)^2$

(ii) $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}$

$= \left(\frac{a}{b}\right)^2 - 2\left(\frac{a}{b}\right)\frac{b}{a} + \left(\frac{b}{a}\right)^2$

$= \left(\frac{a}{b} - \frac{b}{a}\right)^2$

(iii) $(x + y)^2 - 14z(x + y) + 49z^2$

$= (x + y)^2 - 2(x + y)(7z) + (7z)^2$

$= (x + y - 7z)^2$

(iv) $12x^2 - 36x + 27$

$= 3(4x^2 - 12x + 9)$

$= 3[(2x)^2 - 2(2x)(3) + (3)^2$

$= 3(2x - 3)^3$

Q4. (i) $3x^2 - 75y^2$

$= 3(x^2 - 25y^2)$

$= 3[(x^2) - (5y)]$

$= 3(x + 5y)(x - 5y)$

(ii) $x(x - 1) - y(y - 1)$

$= x^2 - x - y^2 + y$

$= x^2 - y^2 - x + y$

$= (x + y)(x - y) - 1(x - y)$

$= (x - y)(x + y - 1)$

(iii) $128am^2 - 242an^2$

$= 2a(64m^2 - 121n^2)$

$= 2a\{(8m)^2 - (11n)^2\}$

$= 21(8m + 11n)(8m - 11n)$

(iv) $3x - 243x^3$

$= 3x(1 - 81x^2)$

$= 3x\{(1)^2 - (9x)^2\}$

$= 3x(1 + 9x)(1 - 9x)$

Q5. (i) $x^2 - y^2 - 6y - 9$

$= x^2 - (y^2 + 6y + 9)$

$= x^2 - [(y^2) + 2(y)(3) + (3)^2]$

$= x^2 - (y + 3)^2$

$= [x + (y + 3)]\ [x - (y + 3)]$
$= (x + y + 3)\ (x - y - 3)$

(ii) $\boldsymbol{x^2 - a^2 + 2a - 1}$
$= x^2 - (a^2 - 2a + 1)$
$= x^2 - [(a)^2 - 2(a)\ (1) + (1)^2]$
$= x^2 - (a - 1)^2$
$= (x)^2 - (a - 1)^2$
$= [x + (a - 1)]\ [x - (a + 1)]$
$= (x + a - 1)\ (x - a + 1)$

(iii) $\boldsymbol{4x^2 - y^2 - 4x - 2y + 3}$
$= 4x^2 - (y^2 + 2y + 1)$
$= (2x)^2 - (y + 1)^2$
$= [2x + (y + 1)]\ [2x - (y + 1)]$
$= (2x + y - 1)\ (2x - y - 1)$

(iv) $\boldsymbol{x^2 - y^2 - 4x - 2y + 3}$
$= x^2 - 4x - y^2 - 2y + 3$
$= x^2 - 4x - y^2 - 2y + 3$
$= x^2 - 4x + 4 - y^2 - 2y + 1$
$= x^2 - 4x + 4 - (y + 1)^2$
$= [(x - 2) + (y + 1)]\ [(x - 2) - (y + 1)]$
$= (x - 2 + y + 1)\ (x - 2 - y - 1)$
$= (x + y - 1)\ (x - y - 3)$

(v) $\boldsymbol{25x^2 - 10x + 1 - 36z^2}$
$= (5x)^2 - 2(5x)(1) + (1)^2 - 36z^2$
$= (5x - 1)^2 - (6z)^2$
$= [(5x - 1) + 6z]\ [(5x - 1) - 6z]$
$= (5x - 1 + 6z)(5x - 1 - 6z)$
$= (5x + 6z - 1)(5x - 6z - 1)$

(vi) $\boldsymbol{x^2 - y^2 - 4xz + 4z^2}$
$= x^2 - 4xz + 4z^2 - y^2$
$= (x)^2 - 2(x)(2z) + (2z)^2 - y^2$
$= (x - 2z)^2) - (y)^2$
$= [(x - 2z)^2 + y]\ [(x - 2z)\ y]$
$= (x - 2z + y)\ (x - y - 2z)$
$= (x + y - 2z)\ (x - y - 2z)$

EXERCISE 5.2

Q1. Factorize

(i) $x^4 + \frac{1}{x^4} - 3$

Solution:

$$= x^4 + \frac{1}{x^4} - 2 - 1 = x^4 - 2 + \frac{1}{x^4} - 1$$
$$= \left(x^2 - \frac{1}{x^2}\right)^2 - 1^2$$
$$= \left[\left(x^2 - \frac{1}{x^2}\right) + 1\right]\left[\left(x^2 - \frac{1}{x^2}\right) - 1\right]$$
$$= \left(x^2 - \frac{1}{x^2} + 1\right)\left(x^2 - \frac{1}{x^2} - 1\right)$$

(ii) $3x^4 + 12y^4$

Solution:

$$= 3(x^4 + 4y^4)$$
$$= 3(x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2)$$
$$= 3(x^2 + 2y^2)^2 - 4x^2y^2$$
$$= 3[(x^2 + 2y^2) - (2xy)^2]$$
$$= 3[x^2 + 2y^2) + 2xy]\,[(x^2 + 2y^2) - 2xy]$$
$$= 3(x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$$

(iii) $a^4 + 3a^2 b^2 + 4b^4$

Solution:

$$= a^4 + 4a^2b^2 + 4b^2 - a^2b^2$$
$$= (a^2 + 2b^2)^2 - (ab)^2$$
$$= (a^2 + 2b^2 + ab)\,(a^2 + 2b^2 - ab)$$
$$= (a^2 + ab + 2b^2)\,(a^2 - ab + 2b^2)$$

(iv) $4x^4 + 81$

Solution:

$$= (2x^2)^2 + (9)^2 + 36x - 36x^2$$
$$= (2x^2 + 9)^2 - (6x)^2$$
$$= (2x^2 + 9 + 6x)\,(2x^2 + 9 - 6x)$$
$$= (2x^2 + 6x + 9)\,(2x^2 - 6x + 9)$$

(v) $x^4 + x^2 + 25$

Solution:

$$= x^4 + 10x^2 + 25 - 9x^2$$
$$= (x^2)^2 + 2(x^2)5 - 9x^2$$
$$= (x^2+5)^2 - (3x)^2$$

$= (x^2 + 5 + 3x)(x^2 + 5 - 3x)$
$= (x^2 + 2x + 4)(x^2 - 2x + 4)$

(vi) $x^4 + 4x^2 + 16$

Solution:

$= x^4 + 8x^2 + 16 - 4x^2$
$= x^4 + 8x^2 + 16 - 4x^2$
$= (x^2 + 4)^2 - (2x)^2$
$= (x^2 + 4 + 2x)(x^2 + 4 - 2x)$
$= (x^2 + 2x + 4)(x^2 - 2x + 4)$

Q2. **(i)** $x^2 + 14x + 48$

Solution:

$= x^2 + 8x + 6x + 48$
$= x(x + 8) + 6(x + 8)$
$= (x + 8)(x + 6)$

(ii) $x^2 - 21x + 108$

Solution:

$= x^2 - 12x - 9x + 108$
$= x(x - 12) - 9(x - 12)$
$= (x - 12)(x - 9)$

(iii) $x^2 - 11x - 42$

Solution:

$= x^2 - 14x + 3x - 42$
$= x(x - 14) + 3(x - 14)$
$= (x - 14)(x + 3)$

(iv) $x^2 + x - 132$

Solution:

$= x^2 - 12x - 11x - 132$
$= x(x + 2)(x - 11)$
$= (x + 12)(2x + 1)$

Q3. **(i)** $4x^2 + 12x + 5$

Solution:

$= 4x^2 + 10x + 2x + 5$
$= 2x(2x + 5) + 1(2x + 5)$
$= (2x + 5)(2x + 1)$

(ii) $30x^2 + 7x - 15$

Solution:

$= 30x^2 + 25x - 18x - 15$
$= 5x(6x + 5) - 3(6x + 5)$

$= (6x + 5)(5x - 3)$

(iii) $\mathbf{24x^2 - 65x + 21}$

Solution:

$= 24x^2 - 56x - 9x + 21$

$= 8x(3x - 7) - 3(3x - 7)$

$= (3x - 7)(8x - 3)$

(iv) $\mathbf{5x^2 - 16x - 21}$

Solution:

$= 5x^2 - 21x + 5x - 21$

$= x(5x - 21) + 1(5x - 21)$

$= (5x - 21)(x + 1)$

(v) $\mathbf{4x^2 - 17xy + 4y^2}$

Solution:

$= 4x^2 - 16xy - xy + 4y^2$

$= 4x(x - 4y) - y(x - 4y)$

$= (x - 4y)(4x - y)$

(vi) $\mathbf{3x^2 - 38xy - 13y^2}$

Solution:

$= 3x^2 - 39xy + xy - 13y^2$

$= 3x(x - 13y) + y(x - 13y)$

$= (x + 13y)(3x - 2y)$

(vii) $\mathbf{5x^2 + 33xy - 14y^2}$

Solution:

$= 5x^2 + 35xy - 2xy - 14y^2$

$= 5x(x + 7y) - 2y(x + 7y)$

$= (x + 7y)(5x - 2y)$

(viii) $\left(5x - \frac{1}{x}\right)^2 + 4\left(5x - \frac{1}{x}\right) + 4$

Solution:

Let $5x - \frac{1}{x} = y$

$= y^2 + 4y + 4$

$= (y + 2)^2 = (y + 2)(y + 2)$

By putting value of $y = 5x - \frac{1}{x}$

$= \left(5x - \frac{1}{x} + 2\right)\left(5x - \frac{1}{x} + 2\right)$

Q4. (i) $\left(x^2 + 5x + 4\right)\left(x^2 + 5x + 6\right) - 3$

Solution:

Let $x^2 + 5x = y$

$$(y+4)(y+6)-3$$
$$= y^2+6y+4y+24-3$$
$$= y^2+10y+21$$
$$= y^2+7y+3y+21$$
$$= y(y+7)+3(y+7)$$
$$= (y+7)(y+3)$$

By putting value of $y = x^2+5x$

$$= (x^2+5x+7)(x^2+5x+3)$$

(ii) $(x^2-4x)(x^2-4x-1)-20$

Solution:

Let $x^2-4x = y$

$$= y(y-1)-20$$
$$= y^2-y-20$$
$$= y^2-5y+4y-20$$
$$= y(y-5)+4(y-5)$$
$$= (y-5)(y+4)$$

By putting value of $y = x^2-4x$

$$= (x^2-4x-5)(x^2-4x+4)$$
$$= (x^2-5x+x-5)[x^2-2(x)2+4]$$
$$= [(x(x-5)+1(x-5)]\,(x-2)^2$$
$$= (x-5)(x+1)(x-2)^2$$

(iii) $(x+2)(x+3)(x+4)(x+5)-15$

Solution:

By using commutative property of addition

As $2+5 = 3+4$

$$= (x+2)(x+5)(x+3)(x+4)-15$$
$$= (x^2+7x+10)(x^2+7x+12)-15$$

Let $x^2+7x = y$

$$= (y+10)(y+12)-15$$
$$= y^2+22y+120-15$$
$$= y^2+22y+120-15$$
$$= y^2+22y+105$$
$$= y^2+15y+7y+105$$
$$= y(y+15)+7(y+15)$$
$$= (y+15)(y+7)$$

By putting value of $y = x^2+7x$

$$= (x^2+7x+15)(x^2+7x+7)$$

(iv) $(x+4)(x-5)(x+6)(x-7)-504$

Solution:

By using commutative property of subtraction

As $4-5=6-7$

$=(x^2-x-20)\ (x^2-x-42)-504$

Let $x^2-x=y$

$=(y-20)\ (y-42)-504$

$=y^2-42y-20y+840-504$

$=y^2-62y+336$

$=y^2-56-6y+336$

$=y(y-56)(y-6)$

By putting value of $y=x^2-x$

$=(x^2-x-56)(x^2-x-6)$

$=x^2-8x+7x-56)\ (x^2-3x+2x-6)$

$=[x(x-8)+7(x-8)][x(x-3)+2(x-3)]$

$=(x-8)\ (x+7)\ (x-3)\ (x+2)$

(v) $(x+1)(x+2)(x+3)(x+6)-3x^2$

Solution:

By using commutative property of multiplication

As $1\times 6=2\times 3$

$=(x+1)\ (x+6)\ (x+2)\ (x+3)-3x^2$

$=(x^2+7x+6)\ (x^2+5x+6)-3x^2$

$=(x^2+6+7x)\ (x^2+6+5x)-3x^2$

Let $x^2+6=y$

$=(y+7x)\ (y+5x)-3x^2$

$=y^2+5xy+7xy+35x^2-3x^2$

$=y^2+12xy+32x^2$

$=y^2+8xy+4xy+32x^2$

$=y(y+8x)+4x(y+8x)$

$=(y+8x)(y+4x)$

By putting value of $y=x^2+6$

$=(x^2+6+8x)(x^2+6+4x)$

$=(x^2+8x+6)(x^2+4x+6)$

$=x\left(x+8+\frac{6}{x}\right)\ .\ x\left(x+4+\frac{6}{x}\right)$

$=x^2\left(x+\frac{6}{x}+8\right)\left(x+\frac{6}{x}+4\right)$

Q5. (i) $x^3 + 48x - 12x^2 - 64$

Solution:

$$= x^3 - 12x^2 + 48x - 64$$
$$= x^3 - 3.x^2.4 + 3.x.4^2 - 4^3$$
$$= (x-4)^3$$

(ii) $8x^3 + 60x^2 + 150x + 125$

Solution:

$$= (2x)^3 + 3.(2x)^2.5 + 3.(2x).5^2 + 5^3$$
$$= (2x-5y)^3$$

(iii) $x^3 - 18x^2 + 108x - 216$

Solution:

$$= x^3 - 3x^2.6 + 3.x.6^2 - 6^3$$
$$= (x-6)^3$$

(iv) $8x^3 - 125y^3 - 60x^2y + 150xy^2$

Solution:

$$= 8x^3 - 60x^2y + 150xy^2 - 125y^3$$
$$= (2x)^3 - 3.(2x)^2.5y + 3.(2x).(5y)^2 - (5y)^3$$
$$= (2x-5y)^3$$

Q6. (i) $27 + 8x^3$

Solution:

$$= (3)^2 + (2x)^3$$
$$= (3+2x)[3^2 - 3.2x + 2(x)^2]$$
$$= (3+2x)(9 - 6x + 4x^2)$$

(ii) $125x^3 - 216y^3$

Solution:

$$= (5x)^3 - (6y)^3$$
$$= (5x-6y)\,[(5x)^2 + 5x.6y + (6y)^2]$$
$$= (5x-6y)(25x^2 + 30xy + 36y^2)$$

(iii) $64x^3 + 27y^3$

Solution:

$$= (4x)^3 + (3y)^2$$
$$= (4x+3y)\,[(4x)^2 - 4x.3y + (3y)^2]$$
$$= (4x+3y)(16x^2 - 12xy + 9y^2)$$

(iv) $8x^3 + 125y^3$

Solution:

$$= (2x)^3 + (5y)^3$$
$$= (2x+5y)\,[(4x)^2 - 4x.3y + (3y)^2]$$
$$= (2x+5y)(4x^2 - 10xy + 25y^2)$$

EXERCISE 5.3

Q1. Use Remainder theorem to find the remainder when

(i) $3x^3 - 10x^2 + 13x - 6$ **is divided by** $(x-2)$

Solution:

Let $p(x) = 3x^3 - 10x^2 + 13x - 6$

When $p(x)$ is divided by $x - 2$

The remainder $R = p(2)$

$p(2) = 3(2)^3 - 10(2)^2 + 13(2) - 6$

$p(2) = 24 - 40 + 26 - 6 \quad = 4$

Therefore remainder = 4

(ii) $4x^3 - 4x + 3$ **is divided by** $(2x-1)$

Solution:

Let $p(x) = 4x^3 - 4x + 3$

When $p(x)$ is divided by $2x - 1$

Then remainder $R = p\left(\frac{1}{2}\right)$

$p\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right) + 3$

$= 4\left(\frac{1}{8}\right) - 4\left(\frac{1}{2}\right) + 3 = \frac{1}{2} - \frac{4}{2} + 3 \quad = \frac{1-4+6}{2} = \frac{3}{2}$

Therefore remainder = $\frac{3}{2}$

(iii) $6x^4 + 2x^3 - x + 2$ **is divided by** $(x+2)$

Solution:

Let $p(x) = 6x^4 + 2x^3 - x + 2$

When $p(x)$ is divided by $x + 2$

The remainder $R = p(-2)$

$p(-2) = 6(-2)^4 + 2(-2)^3 - 2 + 2$

$= 96 - 16 + 2 + 2 \quad = 84$

Therefore remainder = 84

(iv) $p(x) = (2x+1)^3 + 6(3+4x)^2 - 10$ **is divided by** $(2x+1)$

Solution:

Let $p(x) = (2x+1)^3 + 6(3+4x)^2 - 10$

When $p(x)$ is divided by $2x + 1$

The remainder $R = p\left(-\frac{1}{2}\right)$

$$= \left[2\left(-\tfrac{1}{2}\right) - 1\right]^3 + 6\left[3 + 4\left(-\tfrac{1}{2}\right)\right] - 10$$
$$= (-1-1)^3 + 6(3-2)^2 - 10$$
$$= (-2)^3 + 6(1)^2 - 10$$
$$= -8 + 6 - 10 = -12$$

Therefore remainder $= -12$

(v) $\quad$ **$x^3 - 3x^2 + 4x - 14$ is divided by $(x+2)$**

Solution:

Let $\quad p(x) = x^3 - 3x^2 + 4x - 14$

When $\; p(x)$ is divided by $x + 2$

The remainder $\quad R = p(-2)$

$$= (-2)^3 - 3(-2)^2 + 4(-2) - 14$$
$$= -8 - 12 - 8 - 14 = -42$$

Therefore remainder $= -42$

Q2. (i) $\quad$ **If $(x+2)$ is a factor of $x^2 - 4kx - 4k^2$, then find the value(s) of k.**

Solution:

Let $\quad p(x) = 3x^2 - 4kx - 4k^2$

As $\quad x + 2 = x - (-2)$ is a factor of $p(x)$

So $\quad p(-2) = 0$

$$3(-2)^2 - 4k(-2) - 4k^2 = 0$$
$$12 + 8k - 4k^2 = 0$$

Or $\quad 3 + 2k - k^2 = 0$

$$3 + 3k - k - k^2 = 0$$
$$3(1+k) - k(1+k) = 0$$
$$(1+k)\;(3-k) = 0$$

$1 + k = 0 \quad ; \quad 3 - k = 0$

$k = -1 \quad ; \quad k = 3$

$\Rightarrow \quad k = -1,\ 3$

(ii) $\quad$ **If $(x-1)$ is a factor of $x^3 - kx^2 + 11x - 6$, then find the value(s) of k.**

Solution:

Let $\quad p(x) = x^3 - kx^2 + 11x - 6$

As $\quad x - 1$ is a factor of $p(x)$ we have

$$p(1) = 0$$

i.e. $\quad (1)^3 - k(1)^2 + 11(1) - 6 = 0$

$$1 - k + 11 - 6 = 0$$
$$-k + 6 = 0$$

$\Rightarrow \quad k = 6$

Q3. **Without actual long division determine whether**

(i) $(x-2)$ **and** $(x-3)$ **are factors of**

$p(x) = x^3 - 12x^2 + 44x - 48$

Solution:

$p(x) = x^3 - 12x^2 + 44x - 48$

The remainder for $x - 2$ is

$p(2) = (2)^3 - 12(2)^2 + 44(2) - 48$

$= 8 - 48 + 88 - 48 = 0$

Since remainder = 0, therefore $x - 2$ is a factor of $p(x)$

The remainder for $x - 3$ is

$p(3) = (3)^3 + 2(4)^2 + 44(3) - 48$

$= 27 - 108 + 132 - 48 \quad = 3 \neq 0$

Since remainder $\neq 0$, therefore $x - 3$ is not a factor of $p(x)$

(ii) $(x-2), (x+3)$ **and** $(x-4)$ **are factors of**

$q(x) = x^3 + 2x^2 - 5x - 6$

Solution:

$q(x) = x^3 + 2x^2 - 5x - 6$

The remainder for $x - 2$ is

$p(2) = (2)^3 + 2(2)^2 - 5(2) - 6$

$= 8 + 8 - 10 - 6 = 0$

Since remainder = 0, therefore $x - 2$ is a factor of $q(x)$

The remainder for $x - 4$ is

$p(4) = (4)^3 + 2(4)^2 - 5(4) - 6$

$= 64 + 32 - 20 - 6 = 70 \neq 0$

Since remainder $\neq 0$, therefore $x - 4$ is not a factor of $q(x)$

Q4. **For what value of** m **is the polynomial** $p(x) = 4x^3 - 7x^2 + 6x - 3m$ **exactly divisible by** $(x+2)$

Solution:

$p(x) = 4x^3 - 7x^2 + 6x - 3m$

As p(x) is exactly divisible by x + 2 therefore reminder = 0

i.e. $4(-2)^3 - 7(-2)^2 + 6(-2) - 3m = 0$

$-32 - 28 - 12 - 3m = 0$

$-72 - 3m = 0$

Or $-24 - m = 0$

$\Rightarrow m = -24$

Q5. Determine the value of k if $p(x) = kx^3 + 4x^2 + 3x - 4$ and $q(x) = x^3 - 4x + k$ leaves the same remainder when divided by $(x - 3)$

Solution:

$$p(x) = kx^3 + 4x^2 + 3x - 4$$

When $p(x)$ is divided by $x - 3$, then the remainder $p(3) = 0$

$$p(3) = k(3)^3 + 4(3)^2 + 3(3) - 4$$
$$= 27k + 36 + 9 - 4 \qquad = 27k + 41$$
$$q(x) \quad = x^3 - 4x + k$$

When $q(x)$ is divided by $x - 3$ then the remainder $q(3) = 0$

$$q(3) = (3)^2 - 4(3) + k$$
$$= 27 - 12 + k = 15 + k$$

According to given condition

$$p(3) = q(3)$$
$$27k + 41 = 15 + k$$
$$26k = -26 \qquad \Rightarrow \qquad k = -1$$

Q6. The remainder after dividing the polynomial $p(x) = x^3 + ax^2 + 7$ by $(x + 1)$ is $2b$. Calculate the value of a and b if this expression leaves a remainder of $(b + 5)$ on being divided by $(x - 2)$.

Solution:

$$p(x) = x^3 + ax^2 + 7$$

When $p(x)$ is divided by $x + 1$, then the remainder $p(-1) = 0$

$$(p(-1) = (-1)^3 + a(-1)^2 + 7$$
$$= -1 + a + 7 = a + 6$$

As given remainder = 2b

Therefore calculated remainder = given remainder

$$a + 6 = 2b$$

or $\qquad$ a − 2b = −6 $\qquad$ .... (i)

When $p(x)$ is divided by $x - 2$, then the remainder $p(2) = 0$

$$p(2) = (2)^3 + a(2)^2 + 7 = 8 + 4a + 7 = 4a + 15$$

As given remainder = $b + 5$

Therefore calculated remainder = given remainder

$$4a + 15 = b + 5$$

or $\qquad 4a - b = -10$ $\qquad$ .... (ii)

Multiply eq. (ii) by 2 and subtract from eq. (i), we get

$$a - 2b = -6$$
$$\underline{-8a \mp 2b = \mp 20}$$
$$-7a = 14 \qquad \Rightarrow \qquad a = -2$$

Put $a = -2$ in eq. (i), we get

$$-2 - 2b = -6$$

$\Rightarrow \quad -2b = -4 \quad$ or $\quad b = 2$

So $\quad a = -2, \quad b = 2$

Q7. The polynomial $x^3 + lx^2 + mx + 24$ has factor $(x+4)$ and it leaves a remainder of 36 when divided by $(x-2)$. Find the values of l and m.

Solution:

Let $\quad p(x) = x^3 + lx^2 + mx + 24$

As $\quad x + 4$ is a factor of $p(x)$

i.e. $\quad (-4)^3 + l(-4)^2 + m(-4) + 24 = 0$

$$-64 + 16l - 4m + 24 = 0$$

or $\quad 16l - 4m = 40$

or $\quad 4l - m = 10 \qquad \text{.......(i)}$

When $p(x)$ is divided by $x - 2$

When the remainder is $p(2)$

Then $\quad p(2) = 36$

$$x^3 + lx^2 + mx + 24 = 36$$

$$(2)^3 + l(2)^2 + m(2) + 24 = 36$$

$$8 + 4l + 2m + 24 = 36$$

$$4l + 2m = 4$$

or $\quad 2l + 3m = 2 \qquad \text{........(ii)}$

By adding eq. (i) and eq. (ii), we get

$$6l = 12$$

$\Rightarrow \quad l = 2$

By putting $l = 2$ in eq. (i), we get

$$8 - m = 10$$

$$-m = 2$$

$$m = -2$$

So $\quad l = 2, m = -2$

Q8. The expression $lx^3 + mx^2 - 4$ leaves remainder of -3 and 12 when divided by $(x-1)$ and $(x+2)$ respectively. Calculate the values of l and m.

Solution:

Let $\quad p(x) = lx^3 + mx^2 - 4$

When p(x) is divided by x − 1 the remainder

$p(1) = -3$

i.e. $\quad l(1)^3 + m(1)^2 - 4 = -3$

or $\quad l + m - 4 = -3$

$l + m = 1 \qquad \ldots\ldots\ldots \text{(i)}$

When $p(x)$ is divided by $x + 2$ the remainder

$p(-2) = 12$

or $\quad l(-2)^2 + m(-2)^2 - 4 = 12$

$-8l + 4m - 4 = 12$

or $\quad -8l + 4m = 16$

or $\quad -2l + m = 4 \qquad \ldots\ldots\ldots\ldots \text{(ii)}$

Subtracting eq. (ii) from eq. (i), we get

$3l = -3$

Or $\quad l = -1$

Putting $l = -1$ in eq. (i), we get

$-l + m = 1$

$\Rightarrow \quad m = 2$

So $\quad l = -1, \quad m = 2$

Q9. The expression $ax^3 - 9x^2 + bx + 3a$ is exactly divisible $x^2 - 5x + 6$. Find the values of a and b.

Solution:

Let $\quad p(x) = ax^3 - 9x^2 + bx + 3a$

and $\quad q(x) = x^2 - 5x + 6$

$= x^2 - 3x - 2x + 6$

$= x(x - 3) - 2(x - 3)$

$= (x - 3)(x - 2)$

As $p(x)$ is exactly divisible by $q(x)$. So $p(x)$ is exactly divisible by $x - 2$ and $x - 3$ [$\therefore x = 2$ and $x = 3$]

Hence $\quad p(2) = 0$

And $\quad p(3) = 0$

$p(2) = 2(2)^3 - 9(2)^2 + b(2) + 3a = 0$

$8a - 36 + 2b + 3a = 0$

$11a + 2b = 36$

Now $\quad p(3) = a(3)^3 - 9(3)^2 + b(3) + 3a = 0$

$27a - 81 + 3b + 3a = 0$

$30a + 3b = 81$

$10a + b = 27$

By multiplying eq. (ii) by 2 and subtract from eq. (i), we get

$$\begin{array}{r} 11a + 2b = 36 \\ \pm 20a \pm 2b = \pm 54 \\ \hline -9a = -18 \end{array} \qquad \Rightarrow \qquad a = 2$$

Putting $a = 2$ in eq. (ii) we get

$20 + b = 27 \quad \Rightarrow \quad b = 7$

So $\quad a = 2 \quad$ and $\quad b = 7$

EXERCISE 5.4

Factorize each of the following cubic polynomials by factor theorem.

Q1. $x^3 - 2x^2 - x + 2$

Solution:

Let $\quad p(x) = x^3 - 2x^2 - x + 2$

Possible factors of constant zeros of $p(x)$ are $p = \pm 1, \pm 2$

and possible factors of leading coefficient 1 are $q = \pm 1$

Thus the expected zeros of $p(x)$ are

$$\frac{p}{q} = \pm 1, \pm 2$$

Now $p(1) = (1)^3 - 2(1)^2 - 1 + 2$

$= 1 - 2 - 1 + 2 = 0$

Hence $x = 1$ is a zero of $p(x)$ and therefore $x - 1$ is a factor of $p(x)$

$p(-1) = (-1)^3 - 2(-1)^2 - (-1) + 2$

$= -1 - 2 + 1 + 2 = 0$

Hence $x = -1$ is a zero of $p(x)$ and

Therefore $x - 1$ and $x + 1$ is a factor of $p(x)$

$p(2) = (2)^3 - 2(2)^2 - 2 + 2$

$= 8 - 8 - 2 + 2 = 0$

Hence $x = 2$ is a zero of $p(x)$ and therefore $x - 2$ is a factor of p(x)

Hence required factors are $\quad (x - 1)(x + 1)(x - 2)$

Q2. $x^3 - x^2 - 22x + 40$

Solution:

Let $\quad p(x) = x^3 - x^2 - 22x + 40$

Possible factors of constant term 40 are

$$p = \pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 20, \pm 40$$

and those of leading coefficient 1 are $\quad q = \pm 1$

Thus the possible zeros of $p(x)$

$$\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 20, \pm 40$$

Now $\quad p(1) = 1 - 1 - 22 + 44 = 18 \neq 0$

So $\quad x - 1$ is not a factor of $p(x)$

$$p(-1) = (-1)^3 - (-1)^2 - 22(-1) + 40$$
$$= 8 - 4 - 44 + 40 = 0$$
$$= -1 - 1 + 22 + 40 = 60 \neq 0$$

So $x + 1$ is a not a factor of $p(x)$

$$p(2) = (-2)^3 - (-2)^2 - 22(2) + 40$$
$$= 8 - 4 - 44 + 40 = 0$$

So $x - 2$ is a factor of $p(x)$

$$p(-2) = (-2)^2 - 22(-2) + 40$$
$$= 8 - 4 + 44 + 40 = 72 \neq 0$$

So $x + 2$ is not a factor of $p(x)$

$$p(4) = (4)^3 - (4)^2 - 22(4) + 40$$
$$= 64 - 16 - 88 + 40 = 48 \neq 0$$

So $x - 4$ is a factor of $p(x)$

$$p(-4) = (-4)^3 - (4)^2 - 22(4) + 40$$

So $x + 4$ is not a factor of $p(x)$

$$p(5) = (5)^3 - (5)^2 - 110 + 40$$
$$= 125 - 25 - 110 + 40$$
$$= 30 \neq 0$$

So $x - 5$ is not a factor of $p(x)$

$$p(-5) = (-5)^3 - (-5)^2 - 22(-5) + 40$$
$$= -125 - 25 + 110 + 40 = 0$$

So $x + 5$ is a factor of $p(x)$

Hence required factors are $(x - 2)(x - 4)\,(x + 5)$

Q3. $x^3 - 6x^2 + 3x + 10$

Solution:

Let $p(x) = x^3 - 6x^2 + 3x + 10$

Possible factors of constant term 10 are

$$p = \pm 1, \pm 2, \pm 5, \pm 10$$

and those of leading coefficient 1 are $q \pm 1$

Thus the possible zeros of $p(x)$ are

$$\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 20, \pm 40$$

Now $p(1) = 1 - 1 - 22 + 40 = 18 \neq 0$

So $x - 1$ is not a factor od $p(x)$

$$p(-1) = (-1)^3 - (-1)^2 - 22(-1) + 40$$
$$= 8 - 4 - 44 + 40 = 0$$
$$= -1 - 1 + 22 + 40 = 60 \neq 0$$

So $x + 1$ is not a factor of $p(x)$

$$p(2) = (2)^3 - (2)^2 - 22(2) + 40$$
$$= 8 - 4 - 44 + 40 = 0$$

So $x-2$ is a factor of $p(x)$

$$p(-2)=(-2)^3-(-2)^2-22+40$$
$$=-8-4+44+40=72\neq 0$$

So $x+2$ is not a factor of $p(x)$

$$p(-4)=(-4)^3-(-4)^2-22(-4)+40$$
$$=-64-16+88+40=48\neq 0$$

So $x+4$ is not a factor of $p(x)$

$$p(5)=(5)^3-(5)^2-22(5)+40$$
$$=30\neq 0$$

So $x-5$ is not a factor of $p(x)$

$$p(-5)=(-5)^3-(5)^2-22(-5)+40$$
$$=125-25+110+40=0$$

So $x+5$ is a factor of $p(x)$

Hence required factors are $(x-2)(x-4)(x+5)$

Q4. $x^3+x^2-10x+8$

Solution:

Let $p(x)=x^3+x^2-10x+8$

Possible factors of constant term 8 are $p=\pm1,\pm2,\pm4,\pm8$

and those of leading coefficient 1 are $q=\pm1.$

Thus the expected zeros of $p(x)$ are

$$\frac{p}{q}=\pm1,\pm2,\pm4,\pm8$$

$$p(1)=(1)^3+(1)^2-10(1)+8$$
$$=1+1-10+8=0$$

So $x-1$ is a factor of $p(x)$

$$p(-1)=(-1)^3+(-1)^2-10(-1)+8$$
$$=-1+1+10+8=18\neq 0$$

So $x+1$ is not a factor of $p(x)$

$$p(2)=(2)^3+(2)^2-10(2)+8$$
$$=8+4-20+8=0$$

So $x-2$ is not a factor of $p(x)$

$$p(-2)=(-2)^3+(-2)^2-10(-2)^2-10(-2)+8$$
$$=-8+4-40+20+8=-16\neq 0$$

So $x+2$ is not a factor of $p(x)$

$$p(-4)=(-4)^3+(4)^2-10(4)+8$$
$$=-64+16-40+8=0$$

So $x-4$ is not a factor of $p(x)$

$$p(-4)=(-4)^3+(-4)^2-10(-4)+8$$
$$=-64+16+40+8=0$$

So $x+4$ is a factor of $p(x)$
Hence required factors are $(x-1)(x-2)(x+4)$

Q5. $x^3 - 2x^2 + 5x + 6$

Solution:

Let $p(x) = x^3 - 2x^2 + 5x + 6$

Possible factors of constant term 6 are

$p = \pm1, \pm2, \pm3, \pm6$

and those of leading coefficient 1 are $q = \pm1$

Thus the possible zeros of $p(x)$are

$\frac{p}{q} = \pm1, \pm2, \pm3, \pm6$

$p(1) = (1)^3 - 2(1)^2 - 5(1) + 6$
$= -1 - 2 - 5 + 6 = 8 \neq 0$

So $x-1$ is a factor of $p(x)$

$p(-1) = (-1)^3 - 2(-1)^2 - 5(-1) + 6$
$= -1 - 2 + 5 + 6 = 8 \neq 0$

So $x+1$ is not a factor of $p(x)$

$p(2) = (2)^3 - 2(2)^2 - 5(2) + 6$
$\Rightarrow 8 - 8 - 10 + 6 = -4 \neq 0$

So $x-2$ is not a factor of $p(x)$

$p(-2) = (-2)^3 - 2(2)^2 - 5(-2) + 6$
$= -8 - 8 + 10 + 6 = 0$

So $x+2$ is a factor of $p(x)$

$p(3) = (3)^3 - 2(3)^2 - 5(3) + 6$
$= 27 - 18 - 15 + 6 = 0$

So $x-3$ is a factor of $p(x)$
Hence required factors are $(x-1)(x-3)(x+2)$

Q6. $x^3 + 5x^2 - 2x - 24$

Solution:

Let $p(x) = x^3 + 5x^2 - 2x - 24$

Possible factors of constant term -24 are

$\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$

and those of leading coefficient 1 are $q = \pm1$

Thus possible zeros of $p(x)$are

$\frac{p}{q} = \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$

$p(1) = (1)^3 + 5(1)^2 - 2(-1) - 24$
$= 1 + 5 - 2 - 24 = -20 \neq 0$

So $x-1$ is not a factor of $p(x)$

$p(-1) = (-1)^3 + 5(-1)^2 - 2(-1) - 24$
$= -1 + 5 + 2 - 24 = -18 \neq 0$

So $x + 1$ is not a root of a factor of $p(x)$

$p(2) = (2)^3 + 5(2)^2 - 2(2) - 24$
$= 8 + 20 - 4 - 24 = 0$

So $x - 2$ is a factor of $p(x)$

$p(-2) = (-2)^3 + 5(-2)^2 - 2(-2) - 24$
$= -8 + 20 + 4 - 24 = -8 \neq 0$

So $x + 2$ is not a factor of $p(x)$

$p(3) = (3)^3 + 5(3)^2 - 2(3) - 24$
$= 27 + 45 - 6 - 24 = 42 \neq 0$

So $x - 3$ is not a factor of $p(x)$

$p(-3) = (-3)^3 + 5(-3)^2 - 2(-3) - 24$
$= -27 + 45 + 6 - 24 = 0$

So $(x + 3)$ is a factor of $p(x)$

$p(4) = (4)^3 + 5(4)^2 - 2(4) - 24$
$= 64 + 80 - 8 - 24 = 112 \neq 0$

So $x - 4$ is not a factor of $p(x)$

$p(-4) = (-4) + 5(-4) - 2(-4) - 24$
$= -64 + 80 + 8 - 24 = 0$

So $x + 4$ is a factor of $p(x)$

Hence required factors are $(x - 2)(x - 3)(x + 4)$

Q7. $3x^3 - x^2 - 12x + 4$

Solution:

Let $p(x) = 3x^3 - x^2 - 12x + 4$

Possible factors of the constant term 4 are

$p = \pm 1, \pm 2, \pm 4$

and those of the leading coefficient 3 are $q = \pm 1, \pm 3$

Thus the possible zeros of $p(x)$are

$\pm 1, \pm 2, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 4, \pm\frac{4}{3}$

$p(1) = 3(1)^3 - (1)^2 - 12(1) + 4$
$= 3 - 1 + 12 + 4 = -6 \neq 0$

So $x - 1$ is not a factor of $p(x)$

$p(-1) = 3(-1)^3 - (-1)^2 - 12(-1) + 4$
$= 3 - 1 + 12 + 4 = 12 \neq 0$

So $x + 1$ is a not a zero of $p(x)$

$p(2) = 3(2)^3 - (2)^2 - 12(2) + 4$
$= 24 - 4 - 24 + 4 = 0$

So $x - 2$ is not a zero of $p(x)$

$$p(-2) = 3(-2)^3 - (-2)^2 - 12(-2) - 24$$
$$= -24 - 4 + 4 + 4 = 0$$

So $x + 2$ is a zero of $p(x)$

$$p\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{3}\right) + 4$$
$$= \frac{1}{9} - \frac{1}{9} - 4 + 4 = 0$$

So $3x - 1$ is a zero of $p(x)$

Hence $(x - 2), (x + 2)$ and $(3x - 1)$ are factors of **P(x)**.

Hence required factors are $(x - 2)(x + 2)(3x - 1)$

Q8. $2x^3 + x^2 - 2x - 1$

Solution:

Let $p(x) = 2x^3 + x^2 - 2x - 1$

Possible factors of the constant term-1 are $p = \pm 1$

and those of leading coefficient 2 are $q = \pm 1, \pm 2$

Thus the possible zeros $p(x)$are $\frac{p}{q} = \pm 1, \pm \frac{1}{2}$

$$p(1) = 2(1)^3 + (1)^2 - 2(1) - 1$$
$$= 2 + 1 - 2 - 1 = 0$$

So $x - 1$ is a zero of $p(x)$

$$p(-1) = 2(-1)^3 + (-1)^2 - 2(-1) - 1$$
$$= 2 + 1 + 2 - 1 = 0$$

So $x = -1$ is a zero of $p(x)$

$$p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) - 1$$
$$= -\frac{1}{4} + \frac{1}{4} - 11 = 0$$

So $x = -\frac{1}{2}$ is a zero of $p(x)$

Hence $x + 1$ and $2x + 1$ are factors of $p(x)$

Hence required factors are $(x + 1)(x - 1)(2x + 1)$

REVIEW EXERCISE 5

Q1. Multiple choice questions. Choose the correct answer.

(i) The factors of $x^2 - 5x + 6$ are.....

(a) $x + 1, x - 6$ (b) $x - 2, x - 3$

(c) $x + 6, x - 1$ (d) $x + 2, x + 3$

(ii) **Factors of $8x^3 + 27y^3$ are.......**

(a) $(2x + 3y), (4x^2 + 9y^2)$
(b) $(2x - 3y), (4x^2 - 9y^2)$
(c) $(2x + 3y), (4x^2 - 6xy + 9y^2)$
(d) $(2x - 3y), (4x^2 + 6xy + 9y^2)$

(iii) **Factors of 3^{x^2} $-x-2$ are........**

(a) $(x + 1), (3x - 2)$
(b) $(x + 1), (3x + 2)$
(c) $(x - 1), (3x - 2)$
(d) $(2x - 3y), (4x^2 + 6xy)$

(iv) **Factors of $a^4 - x-$ 2 are ....**

(a) $(a - b), (a + b), (a^2 + 4b^2)$
(b) $(a^2 - 2b^2), (a^2 + 2b^2)$
(c) $(a - b), (a + b), (a^2 - 4b^2)$
(d) $(a - 2b), (a^2 + 2b^2)$

(v) **What will be added to complete the square of $9a^2 - 12ab$?**

(a) $-16b^2$ (b) $16b^2$
(c) $4b^2$ (d) $-4b^2$

(vi) **Find m so that $x^2 + 4x + m$ is a complete square...**

(a) 8 (b) −8
(c) 4 (d) 16

(vii) **Factors of $5x^2 - 17xy - 12y^2$ are.......**

(a) (x + 4y), (5x + 3y)
(b) (x − 4y), (5x − 3y)
(c) (x − 4y), (5x + 3y)
(d) (5x − 4y), (x + 3y)

(viii) **Factors of $27x^3 - \frac{1}{x^3}$**

(a) $\left(3x - \frac{1}{x}\right), \left(9x^2 + 3 + \frac{1}{x^2}\right)$
(b) $\left(3x + \frac{1}{x}\right), \left(9x^2 + 3 + \frac{1}{x^2}\right)$
(c) $\left(3x - \frac{1}{x}\right), \left(9x^2 - 3 + \frac{1}{x^2}\right)$
(d) $\left(3x + \frac{1}{x}\right), \left(9x^2 - 3 + \frac{1}{x^2}\right)$

Answers:

(i) b	**(ii) c**	**(iii) d**	**(iv) b**
(v) c	**(vi) c**	**(vii) c**	**(viii) a**

Q2. Complete items. Fill in the blanks.

(i) x + 5x + 6 = ...

(ii) $4a^2 - 16 =$ ....

(iii) $4a^2 + 4ab +$ (........) is a complete square

(iv) $\frac{x^2}{y^2} - 2 + \frac{y^2}{x^2} =$ ....s

(v) $(x + y)(x^2 - xy + y^2) =$ ....

(vi) Factored form of $x^4 - 16$ is ....

(vii) If $x - 2$ is factor of $p(x) = x^2 + 2kx + 8$, then $k =$ ....

Answers:

(i) $(x + 2)(x + 3)$ **(ii)** $4(a - 2)(a + 2)$

(iii) b^2 **(iv)** $\left(\frac{x}{y} - \frac{y}{x}\right)^2$

(v) $x^3 + y^3$ **(vi)** $(x - 2)(x + 2)(x^2 + 4)$

(vii) -3

Q3. Factorize the following.

(i) $x^2 + 8x + 16 - 4y^2$

Solution:

$$= (x+4)^2 - (2y)^2$$
$$= (x + 4 + 2y)(x + 4 - 2y)$$
$$= (x + 2y + 4)(x - 2y + 4)$$

(ii) $4x^2 - 16y^2$

Solution:

$$= 4(x^2 - 4y^2)$$
$$= 4[(x)^2 - (2y)^2]$$
$$= 4(x + 2y)(x - 2y)$$

(iii) $9x^2 + 27x + 8$

Solution:

$$= 9x^2 + 24x + 3x + 8$$
$$= 3x(3x + 8) + 1(3x + 8)$$
$$= (3x + 8)(3x + 1)$$

(iv) $1 - 64z^3$

Solution:

$$= (1)^3 - (4z)^2$$
$$= (1 - 4z)[(1)^2 + 1(4z) + (4z)^2]$$
$$= (1 - 4z)(1 + 4z + 16z^2)$$

(v) $8x^3 - \frac{1}{27y^3}$

Solution:

$$= (2)^3 - \left(\frac{1}{3y}\right)^3$$
$$= \left(2x - \frac{1}{3y}\right)\left[(2x) + 2x\frac{1}{3y} + \left(\frac{1}{3y}\right)^2\right]$$
$$= \left(2x - \frac{1}{3y}\right)\left(4x^2 + \frac{2x}{3y} + \frac{1}{9y^2}\right)$$

(vi) $2y^2 + 5y - 3$

Solution:

$$= 2y^2 + 6y - y - 3$$
$$= 2y(y+3) - 1(y+3)$$
$$= (y+3)(2y-1)$$

(vii) $x^3 + x^2 - 4x - 4$

Solution:

$$= x^2(x+1) - 4(x+1)$$
$$= (x+1)(x^2 - 4)$$
$$= (x+1)(x^2 - 2^2)$$
$$= (x+1)(x+2)(x-2)$$

(viii) $25m^2n^2 + 10mn + 1$

Solution:

$$= (5mn)^2 + 2(5mn).1 + (1)^2$$
$$= (5mn+1)^2$$

(ix) $1 - 12qp + 36p^2q^2$

Solution:

$$= (1)^2 - 2(1)(6pq) + (6pq)^2$$
$$= (1 - 6pq)^2$$